Pattern Derivatives Extended

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Abstract

In this paper we extend the theory of pattern derivatives which we proposed in a previous paper. The simplest type of pattern derivative is a derivative where the payout is contingent on the occurrence (inclusive) or lack of occurrence (exclusive) of a given pattern in the time series describing the underlying asset. If we simplify the underlying movements using a binomial tree with u(p) and d(down) movements, then a pattern is a sequence of u and d symbols which may appear during the life of the contract. In the previous paper we have proposed an efficient algorithm for pricing pattern derivatives, including those involving more than one pattern. In this paper we provide more examples of applications, extend our results to more complex sets of patterns, and prove that the ordinary derivatives can be reduced to pattern derivatives.

1 Introduction

A *pattern derivative* is a financial instrument, specifically an option, which triggers a payout to the buyer in the event that a pattern in an underlying variable occurs (inclusive) or fails to occur (exclusive) over the life of the derivative [3].

The underlying could be the price of a stock and the pattern-clause could be a set of sequences of up or down movements at finite intervals over some finite period. For example, if the pattern-clause is $\{uddu\}$, and the actual life of the underlying is, *dduuddud*, then the inclusive pattern derivative is *struck* at $t = 7$. As with ordinary derivatives we distinguish between American pattern derivatives, when the payout occurs when a pattern strikes ($t = 7$), and a European variety, when the payout occurs at the end of the term after a pattern has struck ($t = 8$). Conversely, an exclusive pattern derivative with this pattern clause would not payout, because *uddu* did occur.

Pattern derivatives are important both for their theoretical and practical applications. Theoretically, these derivatives can track any underlying to an arbitrary amount of precision. This is because if changes in the underlying price are discretized (binomial approximation) then each possible path of the underlying corresponds to a pattern in the language of pattern derivatives.

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This type of derivatives are important in practice because they provide insurance against specific types of events that cannot be covered by ordinary derivatives.

In the following sections we provide examples of applications, show how ordinary derivatives can be reduced to pattern derivatives and generalize known pricing formula to more complex cases. In particular we extend out previous results to the following cases:

- Mixed pattern derivatives with include both exclusive and inclusive patterns.
- Sequence pattern derivatives which strike when multiple patterns strike in sequence.
- Variable-length pattern derivatives which include patterns of different lengths.
- Patterns with wildcards, i.e. patterns that may include arbitrary sub-patterns.

2 Examples of Application

Consider a grocer who expects to receive ten crates of oranges in every weekly shipment. If he gets exactly ten, he is able to sell them all without trouble. If he gets more than ten, he can still make a profit on any number of crates over ten by applying a discount. However, if the supplier starts to consistently bring less than ten crates every week, the grocer's income is in trouble. In fact, the grocer can quantify that if they bring in less than ten crates a week for seven consecutive weeks, he will incur in a loss of about \$500.

The grocer wants to buy insurance against the pattern described as 7 consecutive shipments below the expected ten crates per week. If we use the symbol u to indicate a shipment equal to or in excess of ten creates, and the symbol d to indicate a shipment short of the expected ten crates, than the pattern that applies is an inclusive $ddddddd$ pattern. Inclusive because the grocer expects the payout of \$500 to compensate him for the occurrence of the pattern.

One way to solve this problem is by brute force approach. If for example the life of the option is n days and if the u probability is the same as the d probability and equal to 1/2 we can easily compute all possible strings of length n and the probability of finding any substring in strings of length n. In the table below we computed the probability of finding patterns of 1 and 2 symbols within strings of length n for different values of n:

n	u	d	uu	ud	du	dd
1	1/2	1/2	0/2	0/2	0/2	0/2
2	3/4	3/4	1/4	1/4	1/4	1/4
3	7/8	7/8	3/8	4/8	4/8	3/8
4	15/16	15/16	8/16	11/16	11/16	8/16
5	31/32	31/32	19/32	26/32	26/32	19/32
6	63/64	63/64	43/64	57/64	57/64	43/64
7	127/128	127/128	94/128	120/128	120/128	94/128
8	255/256	255/256	201/256	247/256	247/256	201/256
9	511/512	511/512	423/512	502/512	502/512	423/512

This approach is computationally expensive and grows exponentially with the size of the pattern. It becomes unfeasible for large patterns. Notice that the pattern dd is less likely than the pattern ud for many values of n. This is counter-intuitive since we assumed that u and d have the same probability of occurring. The reason for the difference is that one of the patterns overlaps with itself and one does not. This is the main complication of this problem and it will be reflected in our analytical solutions.

As another example consider an oil company that owns an extraction well which normally produces about 1M barrels of oil per week, with a standard deviation of 100,000 barrels. The company is relying on this well to produce at the same average levels for the next year. Nevertheless it is possible, that because of technical problems (operational risk) or because of other problem with the extraction basin, that the well starts producing less than 800,000 barrels of oil per week for 2 weeks in a row. For the company this would constitute a loss of \$40M. As in the previous case, if we use the u symbol to represent high production and d for low production (below 800,000 barrels week), they may want to buy an inclusive pattern derivative with a payoff of \$40M if the pattern dd strikes.

Similar considerations apply to using pattern derivatives to hedge against market crashes. Note however, that our results in the following sections apply to patterns in general, not only to the case of runs [WHAT DO YOU MEAN CASE OF RUNS]. For example, a pattern option designed to hedge against a flash crash may make use of a pattern du, for d and u suitably defined. Furthermore, there is no restriction to binary values. Our alphabet may be arbitrarily large, and this in itself adds nothing in terms of computational costs for the formulas we provide.

3 Solutions to the Examples

In ref. [3] we derived general formulas for pricing pattern derivatives.

For the inclusive European digital:

$$C(Q_i, n) = Ae^{-rn/N} \sum_{t=k}^{n} \sum_{x \in Q_i} P(x, t) \tag{1}$$

For the inclusive American digital, as:

$$C(Q_i, n) = A \sum_{t=k}^{n} \sum_{x \in Q_i} P(x, t) e^{-rt/N} \tag{2}$$

For the exclusive European digital:

$$C(Q_e, n) = Ae^{-rn/N}(1 - \sum_{t=k}^{n} \sum_{x \in Q_e} P(x,t)) \quad (3)$$

Here A is the payout, $e^{-rn/N}$ the discount factor, r is the risk free interest rate, N days continuously compounded for the term of n out of N days. Q_i is a list of inclusive patters and Q_e is a list of exclusive patters. x is a generic pattern and $P(x,t)$ is the probability that pattern x occurs exactly at time t and neither x or any pattern in the respective Q occur before. P can be computed using the following recursive expression due to ref. [1]:

$$P(x,t) = \frac{1}{L^k}\left(1 - \sum_{j=k}^{t-k} \sum_{y \in Q} P(y,j) - \sum_{j=t-k+1}^{t-1} \sum_{y \in Q} P(y,j)\epsilon_{k-t+j}(y,x)L^{k-t+j}\right) \quad (4)$$

The coefficient $\frac{1}{L^k}$ is the probability of the pattern occurring at t, where L is the size of the alphabet and each symbol is assumed to occur with equal frequency. This is multiplied by the probability that the pattern-clause has not already been struck prior to t conditional on x occurring at t, i.e. 1 - the probability that the pattern-clause has already been struck prior to t conditional on x occurring at t.

The probability that the pattern-clause has already been struck prior to t conditional on x occurring at t is split into two terms: the first is the probability that a pattern in the pattern-clause Q is struck prior to $t-k$, where k is the length of the pattern; the second term is the probability that the pattern-clause is struck at a time between $t-k+1$ and t, that is, at some time after x began. This second term is only non-zero in the event that there is a pattern in the pattern-clause Q which overlaps with x (including x itself). The function $\epsilon_i(x,y)$, is the overlap indicator which measures if x overlaps with y at index i in y.

Using eq. 2 we can solve, for example the grocer problem of the previous section. It is an American inclusive derivative, with an alphabet of size 2, a pattern *ddddddd*, a term of six months (n=26), and payoff of \$500. Using a discount rate of 5%, we obtain a price for the option of \$40.24; which can be compared to the same derivative but with any pattern of the same length which does not overlap with itself, say *ddddddu*, which has a price of \$75.19. Furthermore, while our initial formulas consider only cases where symbols have the same probability of occurring, we can still use the formula for the oil case. If the event d has only a 5% chance of occurring, while the "other" region u occurs 95% of the time, we can price the derivative using the pattern dd and an alphabet of size 20, with the hypothetical assignment of 19 symbols to the region covered by u (19*5%=95%), the details of which do not matter because u does not appear in our pattern clause. We will briefly discuss the use of distributions other than $\frac{1}{L^k}$ for the probability that a pattern occurs at a given time, in our concluding remarks.

4 Mixed-Pattern Cases

So far our attention has been restricted to those pattern derivatives which do not mix an inclusive and exclusive clause. Furthermore we only supplied the tools for pricing these derivatives at the very beginning of their term. In this section we provide formulas for pricing these derivatives at any point during their term. This allows us to later generalize to the inclusive/exclusive mixed case.

The full generalization of the basic pattern derivative probability formula is given as an extension to eq. 4 in ref. [3], section 6, as:
If $k < t$

$$P(x,t|w) = \frac{1}{L^k}\left(1 - \sum_{j=1}^{t-k}\sum_{y\in Q} P(y,j|w) - \sum_{j=t-k+1}^{t-1}\sum_{y\in Q} P(y,j|w)\epsilon_{k-t+j}(y,x)L^{k-t+j}\right) \tag{5}$$

else if $0 < t <= k$

$$P(x,t|w) = \epsilon_{k-t}(w,x)\frac{1}{L^t}\left(1 - \min\left(1, \sum_{j=1}^{k-1}\sum_{u\in Q}\epsilon_j(u,x)\epsilon_{2*k-(t+j)}(w,u)\right)\right) \tag{6}$$

This equation allows for the specification of w, a prefix to the current string. This prefix allows for the pattern derivative to be valued during the life of the contract, not just at the beginning. For instance, if the contract mentioned in the introduction were to be valued at time $t = 5$, ie. when we already have the series of events $dduud$ in the term, we could account for the fact that ud has already just occurred, and we only need an immediate du to strike the pattern $uddu$. With this, the contract would be worth more than it was two intervals before. Because of these properties, in ref. [3], we refer to this formulas, as the *Market Tradable* formula and it is critically important to our generalization to the Mixed-Pattern case.

The mixed pattern derivative has both an inclusive and an exclusive clause. It pays if a pattern from the inclusive pattern-clause is struck and no pattern in the exclusive pattern-clause is struck. The mixed case comes in both American and European varieties as well. In the American variety, payout is made when an inclusive pattern is struck, provided that no pattern in the exclusive pattern clause has been struck (and is not struck simultaneously). In the European case, an inclusive pattern must be struck at some point in the term, and no exclusive pattern may be struck at any point in the term – so that one must wait until the end of the term for payout and payout is conditional on no exclusive pattern being struck during the wait.

In the American mixed case, our requirement is almost exactly the same as the basic American case, except that the strings in the *exclusive* clause cannot be matched prior to the occurrence of the given inclusive pattern being matched either. This implies that, in eq. 5 and eq. 6, we must set $Q = Q_i \cup Q_e$. Yet, in the valuation formula, we must sum over only Q_i. Assuming Q in $P(x,t|w)$ is defined as $Q = Q_i \cup Q_e$, we obtain:

$$C(n, Q|w) = A \sum_{t=1}^{n} \sum_{y \in Q_i} P(y,t|w) e^{-rt/N} \tag{7}$$

The valuation of the European case is similar, except that Q_e cannot be struck after Q_i is struck, because they payout has already occurred at that time. To account for this, we multiply the probability used above, by one minus the probability of an item from the exclusive set occurring after t, conditional on y in Q_i occurring at t, namely using y as a prefix.

Assuming $Q = Q_i \cup Q_e$ and explicitly labeling the probability function with a pattern-clause V as in $P_V(x,t|w)$ and obtain:

$$C(n, Q|w) = Ae^{-rn/N} \sum_{t=1}^{n} \sum_{y \in Q_i} \left(P_Q(y,t|w) \left(1 - \sum_{j=1}^{n-t} \sum_{x \in Q_e} P_{Q_e}(x,j|y) \right) \right) \tag{8}$$

In all out derivations we assumed that the input is *consistent*, i.e. the exclusive clause does not contain a pattern also in the inclusive clause.

5 Alternative Underlying Models

We have thus far assumed that the underlying allows for a uniform distribution across the letters. This means that at any given point in the underlying

series the probability of getting a given pattern is $\frac{1}{L^k}$. This assumption is simplifying but unnecessary. We can instead use a general function for the probability of a given pattern $pr(p)$. Using this general function, our formula from the prior section becomes:
If $k < t$ $P(x, t|w) =$

$$pr(x) - \sum_{j=1}^{t-k} \sum_{y \in Q} P(y, j|w) pr(y) - \sum_{j=t-k+1}^{t-1} \sum_{y \in Q} P(y, j|w) pr(y[0 : t-j]) \epsilon_{k-t+j}(y, x) \tag{9}$$

else if $0 < t <= k$

$$P(x, t|w) = \epsilon_{k-t}(w, x) pr(x[k-t : k]) \left(1 - \min\left(1, \sum_{j=1}^{k-1} \sum_{u \in Q} \epsilon_j(u, x) \epsilon_{2*k-(t+j)}(w, u)\right)\right) \tag{10}$$

Where we here redistribute the probability to the separate terms, because it may not be uniform across patterns (Note that what matters in the second and third terms is the probability of the patterns $y \in Q$ and not the probability of the pattern x.) We also introduce the convention of using indexing into a pattern p between r and s with $p[r : s]$.

One important instance of $pr(p)$ would be the binomial probability, choosing a suitable risk-free probability for an underlying series of u and d. Using this model, ordinary European derivatives can be theoretically replicated by a portfolio of inclusive pattern derivatives under a discrete approximation, where 1) the length of the term equals the length of the patterns involved (the patterns are maximal), 2) the payoff is the payoff under the binomial model for each path of the underlying, and 3) because maximal patterns are independent, the price of the portfolio is the sum:

$$C_p(X, N) = e^{-r} \sum x \in X A_x * P(x, N) \tag{11}$$

Where the payoff A_x is made a function of the pattern $x \in X$, $P(x, N)$ is our standard probability, and X is the set of maximal patterns (length N paths) which lead to a payoff.

The second and third terms in $P(x, N)$ drop out for maximal patterns, making the mathematics trivial. Moreover it would be impractical to list out all paths with payouts for a standard contract. But this is intended to show that standard derivatives are theoretically a special case of of our framework.

6 Patterns of Variable Length

Given the extension of the previous section to allow for alternative probabilities, we can incorporate patterns of variable length. In this case, using the same uniform distribution we used originally, our $pr(p)$ would simply be $\frac{1}{L^{len(p)}}$ for varying p.

In this case we assume that the input is *consistent* as previously described, but furthermore we require that it is consistent in the sense that exclusive patterns do not include inclusive patterns as sub-patterns and vice versa. If the pattern clauses are not consistent, however, they can be made consistent using the following rules:

- If x is a pattern in Q_i and it occurs in another pattern, y, also in Q_i, then whenever y occurs, x will also occur at an equal or prior time, so remove y (it is redundant).

- If x is a pattern in Q_e and it occurs in another pattern, y, also in Q_e, then whenever y occurs,x will also occur at an equal or prior time, so remove y (it is redundant)

- If x is a pattern in Q_e and it occurs in a pattern y in Q_i, then remove y, since whenever y occurs, x occurs at an equal or prior time, and striking y fails.

- If x is a pattern in Q_i and it occurs in a pattern y in Q_e, then remove y, if x does not occur only at the very end of y. (for American Only).

In summary, for any couple of patterns x and y in $Q_e \cup Q_i$, you can remove y if x occurs in y, unless $x \in Q_i$ and $y \in Q_e$. If x is an inclusive pattern and y is an exclusive pattern and x occurs in y, then if x occurs during the term, you must still determine if y is struck in the future in the European case. In the American case, it can be removed, unless x occurs only at the end of y. The apt handling of the special case where inclusion of a pattern in another pattern cannot be discarded is not addressed by us here.

7 Patterns with Wildcards

Here we introduce methods of evaluating the probability of striking a pattern when the pattern may include wildcards *, which match any symbol in the

alphabet. The basic characteristic of patterns involving wildcards are that the probabilities associated with a pattern involving a wildcard are to be amplified by the *cardinality* of the pattern, namely, the number of strings it specifies. This amplification takes place by subtracting out the appropriate number of wildcards at the correct locations, as indicated with our use of $c(x)$ =the number of wildcards in x, below, to get the following formula:
If $k_x < t, P(x,t|w) =$

$$\frac{1}{L^{k_x - c(x)}} \left(1 - \sum_{j=1}^{t-k_x} \sum_{y \in Q} P(y,j|w) - \sum_{j=t-k_x+1}^{t-1} \sum_{y \in Q} P(y^*,j|w) \epsilon_{k_x-t+j}(y,x) L^{k_x-t+j-c(x[0:k_x-t+j])} \right) \tag{12}$$

else if $0 < t <= k_x, P(x,t|w) =$

$$\epsilon_{k_x-t}(w,x) \frac{1}{L^{t-c(x[k_x-t:k_x])}} \sum_{j=1}^{k_x-1} \sum_{y \in Q} \epsilon_j(y,x) \epsilon_{(k_y+k_x-(t+j))}(w,y) P(y^*, t-(k_x-j)|w) L^{t-(k_x-j)-c(x[k_x-t:j])} \tag{13}$$

We first address the issue of wildcards by altering L^{k_x} from previous formulas, to $L^{k_x-c(x)}$, which amplifies the probability of the string by the number of wildcards (by effectively reducing its length). Note that we introduce here a special notation: $x[i:j]$ = the pattern x from index i to j (one based). This is used in the second case to remove only wildcards which lie in the region which does not overlap with the prefix, with $L^{t-c(x[k_x-t:k_x])}$. It is also used in $L^{k_x-t+j-c(x[0:k_x-t]})$, which we use in place of L^{k_x-t+j}. This had amounted to an amplification of the probabilities in the third term. This amplification was based on the fact that there were $k_x - t - j$ terms from y pre-determined to match, because this was the size of the overlap with x. If there are wildcards in p in the region of overlap these items are now already pre-determined to match and need to be removed from the amplification.

Finally, we use the notation y^* when passing the pattern in the recursive call. This indicates the pattern y, but with those wildcards in y in the region of overlap with x, replaced with the values from x. Without doing this, the recursive call would consider overlaps as valid, which couldn't really exist, for such patterns as, e.g. "bab**aaa".

One must still be careful with the use of wildcards in the context of multi-pattern derivatives. If two patterns are used, which resolve to pattern sets with a non-empty intersection (e.g. "a*bb" and "ab*b"), these formulas will produce a false result, as patterns will be double counted ("abbb") by this method. This is a limitation which would be nice to overcome.

8 Pattern Sequences

In addition to pricing inclusive contracts which rely on the occurrence of a single pattern among a set of patterns, we can also price contracts which rely on a sequence of patterns. Conceptually, this can be done by making the payout of an American pattern derivative another pattern derivative, and so on in a chain of derivatives which end in a payout if the sequence is struck. The formula for this can be simplified to a European variety of pattern sequences to:

$$C(n, S|w) = Ae^{-rn/N} R_S(n|w) \tag{14}$$

$$R_S(n|w) = \sum_{t=1}^{n} (P(S_0, t|w) * R_{S^-}(n - t|S_0) \tag{15}$$

Where here S is a sequence of patterns, S_0 is its first element, and S^- is the remainder (the sequence without its first element). R_{S^-} is a recursive call, passing the sequence S without the first element in the sequence. Notice that R_S is 1 if S is empty and $n > 0$. It is 0 if S is not empty and $n = 0$. Here the relevance of overlap is handled implicitly by including the prior pattern as a prefix in the recursive call. P is our probability function from previous sections.

9 Limitations

All of our formulas involve a disjunction of patterns which may occur in the underlying (in the inclusive case) or a conjunction of patterns which may not (in the exclusive case). These derivatives can be represented as follows:

$$\{I_1 \vee I_2 \vee I_3 \vee ... \vee I_j\} \wedge \{\neg E_1 \wedge \neg E_2 \wedge \neg E_3 \wedge ... \wedge \neg E_i\} \tag{16}$$

With I the inclusive patterns and E the exclusive patterns. It should be mentioned that if you try to include either a conjunction of patterns included

$$\{I_1 \wedge I_2 \wedge I_3 \wedge ... \wedge I_j\} \tag{17}$$

or a disjunction of patterns excluded

$$\{\neg E_1 \vee \neg E_2 \vee \neg E_3 \vee ... \vee \neg E_i\} \tag{18}$$

then using methods similar to the ones we have been considering, you would run into a combinatorial explosion of cases to consider and the formulas would be inefficient for large pattern clauses.

In the case of sequences of patterns (inclusive) we were able to skirt this issue because the order in which the patterns occur is fixed. We were thereby able to make n pattern overlap comparisons, namely, the first pattern with the second, the second with the third, etc. If we did not specify a sequence, but rather a conjunctive unordered set, we would have to make all possible overlap comparisons (e.g. the third pattern with the first pattern and from both the left and the right). We would have similar problems with disjunctions of exclusions.

This seems to be an inherent limitation in pricing pattern derivatives analytically. Though incidentally, you can price the disjunction:

$$\{I_1 \vee I_2 \vee I_3 \vee ... \vee I_j\} \vee \{\neg E_1 \wedge \neg E_2 \wedge \neg E_3 \wedge ... \wedge \neg E_i\} \tag{19}$$

In order to do so, you can use $1 - P$ and substitute the inclusive patterns for the exclusive patterns and visa versa, within our mixed formula.

10 Conclusions

Pattern derivatives form a very general class of derivatives which include standard options as a subclass. In this paper we provided some examples of applications and also extended the theory of pattern derivatives. In particular, we generalized previously known pricing formulas to the mixed-pattern case, pattern sequences, alternative underlying probability models, the case of patterns of variable length, and the case of patterns with simple wildcards.

From here, there are two clear directions we can take. The first direction is accounting for the risk involved in holding portfolios of pattern derivatives, which given possible overlap among the clauses between the pattern derivatives derivatives held, is not typically reflected directly in the prices of the derivatives themselves (the portfolio replicating the standard European Option was an exception). Generally, the mean value of a portfolio is a linear combination of the derivatives in the portfolio, but the sigma is greater if the conditions between the derivatives overlap. Overlap can be avoided by the method already explained, but this may not always be an option. The limitation mentioned in the prior section also appears likely to be a limitation here, in assessing the risk of portfolios of pattern derivatives, as they essentially involve conjunctions of patterns.

The second is specifying more powerful additions to the language for defining useful and more complex pattern derivatives. With respect to the latter, we have made some strides here, but at the forefront is the general problem of providing the theoretically ideal language for the specification of pattern derivatives; one which is maximally concise, expressive (amenable to specifying many and various explicit pattern derivatives) and efficient for pricing. Specifying such a language and proving its optimality (to some degree of compromise) are the next step in the theoretical advancement of Pattern Derivatives.

[TODO: ADD MORE BIBLIOGRAPHY]

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