Pattern derivatives

Casey S. Schroeder* and Massimo Di Pierro

School of Computing,
DePaul University,
CDM Center, 243 South Wabash Avenue,
Chicago, IL, USA
E-mail: cs@csschroeder.com
E-mail: mdipierro@cs.depaul.edu
*Corresponding author

Abstract: In this paper, we propose a new type of derivative called a pattern derivative. In the simple case of an asset that moves up u or down d in value, a pattern is a sequence of $\{u, d\}$ movements that may occur before expiration. We provide general pricing formulas that rely on a brute force approach as well as efficient valuation algorithms based on recursive formulas for the probability of patterns to occur. We generalise our results to exclusive, inclusive, and multi-pattern options. Finally, we discuss the possible benefits of this type of options.

Keywords: patterns; derivatives; pattern derivatives; options; pricing; options pricing.

Reference to this paper should be made as follows: Schroeder, C.S. and Di Pierro, M. (2011) 'Pattern derivatives', *Int. J. Financial Markets and Derivatives*, Vol. 2, No. 4, pp.249–257.

Biographical notes: Casey S. Schroeder is Researcher and Consultant with interests in the fields of rational decision making and leveraging technology for better decision making. He obtained his Master in Philosophy from The University of California at San Diego and Master of Science in Computational Finance from DePaul University in Chicago.

Massimo Di Pierro is an Associate Professor at the School of Computing and Digital Media of DePaul University. He is Director of the Master of Science in Computational Finance and expert in large scale numerical applications of computing in physics and finance. He is the author of many open source libraries including web2py.com and fermiqcd.net. He obtained his PhD in High Energy Particle Physics from the University of Southampton in UK.

1 Introduction

In this paper, we propose a new type of path-dependent derivative, which we call *pattern derivatives*. Pattern derivatives are a family of derivatives characterised by the fact that payouts are dependent upon patterns of sequential, discrete events occurring or failing to occur during the life of the contract.

By a pattern, we mean a string of symbols from an arbitrary but finite alphabet. For example, *bac* is a pattern from the English alphabet and it occurs in the words 'abacus' and 'backward'. Although we will try to be as general as possible, our numerical examples will focus on an alphabet of two symbols $\{u, d\}$ where u may represent a positive daily return and d a negative daily return for an asset (for example a stock, a portfolio, or an index). Patterns such as *dddd*, *uuuu*, *ududud* and *ddudu* will represent patterns of periodic returns on an underlying asset. Most of our results are general and not dependent on this choice of an alphabet.

One particular kind of pattern derivative is a very simple digital option which triggers a one time payout to the holder of the option in the event that the underlying goes down for four consecutive market days (pattern *dddd*) during the life of the contract. Supposing that there are thirty market days until the expiry of the contract, the underlying may have the following description between now and expiration:

*uudduddududuuddd***d***duuudududuudu*

in this case, the bold **d** indicates the point at which the payout is triggered. The payout can be made at any point after the triggering pattern occurs. We can assume, as for a European style option, that the payout is made at the end of the contract.

Patterns can be used instead of a barrier, in any type of barrier option. Pattern derivatives, however, may offer some protection to option writers where barrier options fail. For instance, many down-side barriers on US options were hit during the flash-crash of May 2010, though there was no persistence under these barriers. One could instead write in the contract that these barriers must be breached at close. But should such a crash happen at the end-of-day, where we assume that the markets would again be quickly buoyed by media and government intervention, such conditions would provide no such protection. A pattern option or hybrid pattern-barrier option specifying extended negative returns may help alleviate these risks to writers, while keeping to the spirit of the agreement.

Pattern derivatives are generally very useful in handling known problematic cases. Should a financial company be aware that a certain pattern of returns across one or more holdings would cause significant trouble, they can purchase a pattern derivative to cover this specific case. Furthermore these derivatives are quite useful in cases where the underlying is naturally discrete. For example, while it may be the case that the Fed can change interest rates by any continuous value they would like, one may have good reason to think that they will continue to make their changes in intervals of a quarter of a point. This case is quite amenable to treatment with pattern derivatives, with letters representing quarters of a point change. Finally, and perhaps most importantly, this methodology clearly applies outside of the specifically financial realm, to that of insurance and real options as well. Consider, for example, the case of being dependant on a supplier unable to deliver for many consecutive days, resulting in a shortage of supplies.

In the following sections, we will provide an explanation of patterns and their properties. We explain why these apparently elementary derivatives have somewhat counterintuitive valuations and we will provide details of the calculations which should be used in their valuation. We then make strides towards a fully general treatment and discuss possible extensions and further generalisations.

2 Numerical considerations

To be clear, we need to distinguish between the *inclusive* and *exclusive* use of patterns. Inclusive patterns are patterns that are required to appear during the life of the option (as with *dddd*), and exclusive patterns are patterns that are required not to appear. To start, we will concentrate on inclusive patterns and an alphabet {*u*, *d*} where each symbol has equal probability. We will indicate with *L* the size of our alphabet; in the case of {*u*, *d*}, *L* = 2. We define *P*(*x*, *n*) as the probability of a pattern *x* occurring in a random sequence of length *n* (from our alphabet). Here are some numerical results for the *P*(*x*, *n*) from a brute force approach:

n	*u*	*d*	*uu*	*ud*	*du*	*dd*
1	1/2	1/2	0/2	0/2	0/2	0/2
2	3/4	3/4	1/4	1/4	1/4	1/4
3	7/8	7/8	3/8	4/8	4/8	3/8
4	15/16	15/16	8/16	11/16	11/16	8/16
5	31/32	31/32	19/32	26/32	26/32	19/32
6	63/64	63/64	43/64	57/64	57/64	43/64
7	127/128	127/128	94/128	120/128	120/128	94/128
8	255/256	255/256	201/256	247/256	247/256	201/256
9	511/512	511/512	423/512	502/512	502/512	423/512

The first column is the total length of the strings from {*u*, *d*}, that we are generating. The fractions represent the fraction of those strings that contain the corresponding pattern in the table head. For example: 19 out of 32 strings of length 5 contain the pattern *ud*. The first thing to observe, and rather obvious, is that the totals for pattern *u* are the same as for pattern *d* (and in this case, are equivalent to the total number of strings of length *n*, minus the one string containing all *ds* or all *us*, respectively). It is also obvious that, because of symmetry, the number of occurrences of pattern *uu* and *dd* is the same for any given *n* (as with *ud* and *du*). What is not obvious is that the number of occurrences of *uu* is consistently lower than the number of occurrences of *ud*.

Now we consider a market interpretation of the above data, the case of a stock price and its daily returns. Compatibly with Black-Scholes we can assume that daily returns are independent and normally distributed random variables with average zero; therefore the stock has the same probability of going up or down on each day. Under this interpretation of *u* and *d*, the above result tells us that the probability that in the next *n* days the stock will go up for two consecutive days is lower than the probability that it will go up one day and down the second.

Considering now all patterns of three days. The probability of those patterns occurring over a period of *n* days is:

n	*uuu*	*uud*	*udu*	*udd*
1	0/2	0/2	0/2	0/2
2	0/4	0/4	0/4	0/4
3	1/8	1/8	1/8	1/8
4	3/16	4/16	4/16	4/16
5	8/32	12/32	11/32	12/32
6	20/64	31/64	27/64	31/64
7	47/128	74/128	63/128	74/128
8	107/256	168/256	142/256	168/256
9	238/512	369/512	312/512	369/512

Similar to *uu* and *ud* in our prior example, the probability of *uuu* is lower than *uud*, and because of symmetry, the probability of *udd* is the same as *uud*. It is notable, however, that the probability of up-down-up (*udu*) is also lower than the probability of up-up-down (*uud*). These differences make a sizable contribution to the proper valuation of pattern derivatives, and as we shall see, are accountable to the fact that uuu and udu are patterns which overlap with themselves.

3 Pricing digital one-pattern-options

Continuing with our basic case, we can consider a European digital option that pays $A = 1$ dollar at expiration (after n days) if during the life of the option a pattern occurs (an inclusive use). We assume $N = 360$ days/year and a yearly 5% interest rate, r. We can use the following basic formula for pricing our pattern derivatives with one inclusive pattern:

$$C(x,n) = \frac{1}{L^n} \sum_{s \in S_n} A_f(s,x) e^{-rn/N} \tag{1}$$

where $C(x, n)$ is the price of the derivative, x is the pattern, S_n is a set of all possible sequences of u and d of length n and $f(s, x)$ is a function that returns 1 if x appears in s and 0 otherwise. We compute the prices C for different options of different maturity and different patterns as follows:

n	*u*	*d*	*uu*	*ud*	*uuu*	*uud*
1	0.4999	0.4999	0.0000	0.0000	0.0000	0.0000
2	0.7498	0.7498	0.2499	0.2499	0.0000	0.0000
3	0.8746	0.8746	0.3748	0.4998	0.1249	0.1249
4	0.9370	0.9370	0.4997	0.6871	0.1874	0.2499
5	0.9681	0.9681	0.5933	0.8119	0.2498	0.3747
6	0.9836	0.9836	0.6713	0.8899	0.3122	0.4840
7	0.9912	0.9912	0.7337	0.9366	0.3668	0.5776
8	0.9950	0.9950	0.7843	0.9638	0.4175	0.6555
9	0.9968	0.9968	0.8251	0.9792	0.4643	0.7198

And as we expect, fixing an arbitrary n (time until expiry), the shorter the pattern, the more expensive the option; since the shorter the pattern, the less that has to 'go right'. On the other hand, the options involving patterns which overlap with themselves are generally worth less than patterns of the same length and time to expiration, which are *not self-overlapping*.

Similarly, we can price a European call option involving a single exclusive pattern with the formula:

$$C(y, n) = \frac{1}{L^n} \sum_{s \in S_n} A\big(1 - f(s, y)\big) e^{-rn/N} \tag{2}$$

where we use the letter y to indicate an exclusive pattern instead of the letter x which we used for inclusive patterns.

We can also define a US digital (inclusive) pattern option as one that will be exercised (and therefore pays the \$1) the moment the pattern is expressed. For this purpose, we define a function $g(s, x)$ that returns 0 if the pattern x is not in s or the position of the pattern as an integer number if x is contained in s. In this case, the pricing formula for a US digital pattern option is:

$$C(x, n) = \frac{1}{L^n} \sum\nolimits_{s \in S_n} A \min\big(g(s, x), 1\big) e^{-rg(s,x)/N} \tag{3}$$

The difference between European and US options is negligible for ten days but it becomes sizable and the effect we are showing is even greater for US options, because if a pattern is more likely to occur, it is also more likely to occur sooner, and this affects the timing of the payoff.

At this point generalisations are obvious.

While the above formulas are theoretically sound, looping over S_n becomes computationally expensive for very large n and already impractical for $n > 20$, with an alphabet of just two symbols. Analytical results are therefore needed. Many analytical results about probability theory of pattern occurrence can be found in Blom (1982) and Blom and Thorburn (1982) who build off the work of Feller (1968) and others. According to Blom and Thorburn (1982) and their Theorem 2.1, the probability $P(x, t)$ defined as the probability that a pattern x matches for the first time at t in an arbitrary sequence of length n, can be given by the following recursive relation:

$$P(x, t) = \frac{1}{L^k}\left(1 - \sum_{j=k}^{t-k} P(x, j) - \sum_{j=t-k+1}^{t-1} P(x, j)\varepsilon_{k-t+j}(x) L^{k-t+j}\right) \tag{4}$$

Here, k is the length of pattern x. The first part of the equation is just the probability that the pattern occurs at t, whether or not it is the first occurrence. To help determine the probability of it being the first occurrence we must subtract out the second term, which is the probability of it occurring for the first time at a previous, non-overlapping time. If a pattern does not overlap with itself the third term is zero. But if it does, the function $\varepsilon_j(x)$ indicates the amount of the overlap. It returns one just in case the pattern overlaps with itself at an overlap of size j. For instance, for $v = uudduu$, $\varepsilon_1(v) = \varepsilon_2(v) = 1$, and $\varepsilon_j(v) = 0$ for every other j; and for $v = uuuu$, $\varepsilon_j(v) = 1$ for all j, $1 <= j <= 4$.

With this fundamental formula, the valuations for our basic pattern-options become straightforward and we can rewrite equation (1) for inclusive European digital options as:

$$C(x,n) = Ae^{-rn/N} \sum\nolimits_{t=k}^{n} P(x,t) \tag{5}$$

equation (2), for exclusive European digital, as:

$$C(x,n) = Ae^{-rn/N} \left(1 - \sum\nolimits_{t=k}^{n} P(x,t)\right) \tag{6}$$

and equation (3), for inclusive US digital, as:

$$C(x,n) = A \sum\nolimits_{t=k}^{n} \left(P(x,t) e^{-rt/N}\right) \tag{7}$$

Notice that the brute force approach to the computation of equation (1) has a running time of $O(L^n)$ and it becomes unfeasible for $n > 20$. The exact formula in equation (5) instead has a running time of $O(n^3)$ and it is very efficient even for large values of n.

There are three fundamental generalisations to these basic results which must be addressed to achieve sufficient generalisation of the basic case. These are, respectively, the multi-pattern case, the mixed-pattern case, and the market-tradable case.

4 Multi-pattern options

Multi-patterns allow for the possibility that an inclusive-pattern-option (or an exclusive-pattern-option) may involve more than one pattern. That is, the condition of the option is met just in case one or more of the patterns occur (or in the exclusive case, none of the patterns occur). First of all, we observe that patterns may overlap with themselves and with each other, therefore the pricing cannot be a linear function of the pricing of the individual patterns. We will also define Q_e as a set of exclusive patterns and Q_i as a set of inclusive ones.

From the definition of digital option, we generalise equation (1) for multi-pattern inclusive digital as:

$$C\left(Q_i, n\right) = \frac{1}{L^n} \sum\nolimits_{s \in S_n} \theta\left(\sum\nolimits_{x \in Q_i} f(s,x)\right) Ae^{-rn/N} \tag{8}$$

and we generalise equation (2) for multi-pattern exclusive digital as:

$$C\left(Q_e, n\right) = \frac{1}{L^n} \sum\nolimits_{s \in S_n} \left(1 - \theta\left(\sum\nolimits_{y \in Q_e} f(s,y)\right)\right) Ae^{-rn/N} \tag{9}$$

(Here $\theta(\ldots)$ is 1 if the argument if greater than zero, and zero otherwise.) These valuations from brute force are clearly more computationally intensive than the basic case. Thankfully, Blom and Thorburn (1982) provide an analytical result that allows us to write a general formula for the probability of x matching at t and no other prior matches from Q as:

$$P(x,t)=\frac{1}{L^k}\left(1-\sum_{j=k}^{t-k}\sum_{z\in Q}P(z,j)-\sum_{j=t-k+t}^{t-1}\sum_{z\in Q}P(z,j)\varepsilon_{k-t+j}(z,x)L^{k-t+j}\right) \tag{10}$$

Here, the formula is modified only in order to subtract out the probability of the other patterns occurring at a prior time as well. We do this by including summation over all the z in the set of patterns Q, and modifying $\varepsilon_j(z, x)$ to take two parameters, accounting for overlap between a pattern and itself as well as other patterns. It is clear that we can use this formula for the multi-pattern generalisation, in all of the types of options we have as yet considered. It is notable, however, that this only provides a generalisation to the multi-pattern case where the patterns are of constant length k.

Now, we can rewrite equation (8) as:

$$C(Q_i,n)=Ae^{-rn/N}\sum_{t=k}^{n}\sum_{x\in Q_i}P(x,t) \tag{11}$$

and equation (9) as:

$$C(Q_e,n)=Ae^{-rn/N}\left(1-\sum_{t=k}^{n}\sum_{y\in Q_e}P(y,t)\right) \tag{12}$$

5 Mixed-pattern options

Mixed-pattern options allow us to address the possibility that a single pattern-option may have both a multi-inclusive and a multi-exclusive clause. Consider an option that pays a dividend when one of some x inclusive patterns is expressed and none of some y exclusive patterns are expressed:

$$C(Q_i,Q_e,n)=\frac{1}{L^n}\sum_{s\in S_n}\theta\left(\sum_{x\in Q_i}f(s,x)\right)\left(1-\theta\left(\sum_{y\in Q_e}f(s,y)\right)\right)Ae^{-rn/N} \tag{13}$$

This is the most general formula for a digital option including both inclusive and exclusive patterns. This is (potentially) twice as complicated as the basic multi-pattern cases, so we again are in need of analytic results. For the mixed case, however, analytical results are more difficult to come by. We must combine the cases in a valid way, which considers not simply the overlap of an inclusive (exclusive) pattern with another inclusive (exclusive) pattern, but also accounts for the overlap of inclusive (exclusive) patterns with exclusive (inclusive) patterns. We will discuss this case in a following work.

6 Market tradable pattern options

For any options to be market-tradable, we need to address how the valuation of a pattern option may evolve during the life of an option, considering the fact that a relevant pattern may already be imminent. For instance, if the relevant option involves the pattern *uduuud* and the prior four days returns provided *uduu*, we must consider the fact that it is more likely to strike the pattern now than it was four days ago, for the fact that now only two sequential events need to take place. The inclusion of this case is straightforward, although it introduces an additional level of complexity.

If we identify with w the pattern that is already expressed in the recent past, and we define $P(v, t \mid w)$ as the probability of pattern v occurring for the first time at t given the past history w, we can write a recursive formula for P using:

$$P(v,t \mid w) = \varepsilon_{k-t}(w,v)\left(L^{-t} - \sum_{j=1}^{t-1} \sum_{u \in Q} P(u, j \mid w)\varepsilon_j(u,v)L^{-t+j} \right) \quad (14)$$

for $0 < t < k$ and the previous formula (10), with suitable alterations, when $t \geq k$. Should one wish to trade a pattern option mid-term, this formula must be used with w being the previous $k - 1$ events or how many events have already taken place, whichever is less. It is notable that this formula is a generalisation of the multi-pattern case and makes no special considerations precluding the use of mixed-patterns. $P(v, t \mid w)$ can safely be replaced into equation (12) and equation (11).

7 Conclusions and outlook

We have thus far considered a simple coupon payout. It should be clear to the reader that any sort of instrument which can be valued at a future time can be substituted for these coupon payments. As such, the A in our formulas may represent an asset of any variety or it may itself be an option, as found in certain types of barrier options (knock-in options) and forward starting options (Bouzoubaa and Osseiran, 2010). A may also be negative and used in conjunction with standard models of valuation to indicate a loss when a trigger is met (knock-out options) (Bouzoubaa and Osseiran, 2010).

There are furthermore natural changes that can be made to the payout structure, where a coupon payment is made or accrued for each occurrence of the pattern, for instance *ddd*. These can come in two varieties. In the first variety, the payment is made given every occurrence; in this case, a *d* immediately following three *d*'s in a row, would trigger another coupon payment, for the fact that it also has two *d*'s immediately preceding it. In the second variety, the payment is made given every fresh occurrence; in this case, the trigger resets itself at *d*, and awaits three fresh *d*'s in a row. In the first case, we can very easily give an equation for the expected value of the option, using the coupon and the mean number of occurrences per series, if we assume payment is made at the end of the term. The mean number of occurrences per series is simply $(n - k + 1) * L^{-k}$. Given the considerations above, it is sometimes also a surprising fact that this value is the same for all patterns of length k, given time to expiry n, and the size of the alphabet L, regardless of overlap (though here assuming equal probabilities). However, when there is much overlap, patterns will tend to be struck in bunches; this should be clear from the fact that in the case of *ddd*, it only takes one more *d*, and not three, to initiate another payout. So the distribution of payouts will have a considerably greater variance. In the second case, the expected value is considerably less for patterns with high overlap, so these options should be considerably cheaper than those involving non-overlapping counterparts (in the inclusive use) (Chryssaphinou and Papastavridis, 1988).

And finally, there are very few restrictions on the kinds of underlying and partitions which may be assigned to our alphabet. In our construction of the partition of the underlying distribution, the choice of a two letter alphabet was not essential. All of the formulas here mentioned thus far rely on an alphabet of arbitrary size *L*, and as a computational matter, increasing the size of *L* does not effect time complexity to any problematic degree. This allows, for instance, one to slice the distribution of returns in the way that uses patterns of extreme returns in an option, for the sake of handling volatility, for instance. Similarly, it appears that it is not necessary that each letter in the alphabet have equal probability of occurring. And it does not appear to be necessary that the successive letters even be independently distributed. Formulas are given by Robin and Daudin (1999) for the case of using a non-uniform distribution over the letters and for the first-order Markovian case. (Though it bears repeating that the influence of overlap does not disappear.) And finally, it is clear that there is nothing special about using the partition of the probability distribution of a single variable, as in the case of returns we consider here. Should we have a joint probability distribution over two or more variables, then we can partition it in the assignment of letters to events. This more general case has clear applications in the area of *perfect storms*.

We have here only addressed the most basic logical categories of pattern derivatives, but we find that they are fundamental and elegant to handle. We will elaborate in future work.

References

Blom, G. (1982) 'On the mean number of random digits until a given sequence occurs', *Journal of Applied Probability*, March, Vol. 19, No. 1, pp.136–143.

Blom, G. and Thorburn, D. (1982) 'How many random digits are required until given sequences are obtained?', *Journal of Applied Probability*, September, Vol. 19, No. 3, pp.518–531.

Bouzoubaa, M. and Osseiran, A. (2010) *Exotic Options and Hybrids*, Wiley, NY.

Chryssaphinou, O. and Papastavridis, S. (1988) 'A limit theorem for the number of non-overlapping occurrences of a pattern in a sequence of independent trials', *Journal of Applied Probability*, June, Vol. 25, No. 2, pp.428–431.

Feller, W. (1968) *An Introduction to Probability and it's Applications*, Vol. I, 3, Wiley, NY.

Robin, S. and Daudin, J.J. (1999) 'Exact distribution of word occurrences in a random sequence of letters', *Journal of Applied Probability*, March, Vol. 36, No. 1, pp.179–193.